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# $R(C_n, C_n, C_n) \leqslant (4+o(1))\, n$

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It is shown that the Ramsey number $R(C_n, C_n, C_n)$ is bounded from above by $(4+o(1))\, n$. In particular, if $n$ is odd then $R(C_n, C_n, C_n) = (4+o(1))\, n$. © 1999 Academic Press

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## INTRODUCTION

In the paper we are concerned with the asymptotic behaviour of the Ramsey number $R(C_n, C_n, C_n)$. It has been conjectured by Bondy and Erdős [1] (see also [2]) that $R(C_n, C_n, C_n) \leqslant 4n-3$, which would give the correct value of $R(C_n, C_n, C_n)$ in the case when $n$ is odd. Although we are not able to settle this problem, we show that the value of $R(C_n, C_n, C_n)$ does not grow with $n$ much faster than $4n$.

Theorem 1. $R(C_n, C_n, C_n) \leqslant (4+o(1))\, n$.

Since, as we have already mentioned, for odd $n$ one can easily colour edges of the complete graph $K_{4n-4}$ with three colours not creating monochromatic odd cycles of length $n$, the above result gives the correct asymptotic value of $R(C_n, C_n, C_n)$ in the odd case.

Corollary. *If $n$ is odd then $R(C_n, C_n, C_n) = (4+o(1))\, n$.* ∎

The structure of the note goes as follows. First we recall the statement of Szemerédi's Regularity Lemma and present a few simple results on $(\varepsilon, G)$-regular pairs. The net section contains the key ingredient of our argument, Lemma 9, which states that each colouring of "nearly all" edges of the complete graph $K_{4(1+\eta)\, n}$ with three colours leads to a monochromatic odd cycle of length at least $(1+\eta/10)\, n$. The proof of Theorem 1 is given

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in the last part of the note, where we also discuss possible strengthenings and generalizations of this result.

## 1. REGULAR PAIRS AND THE REGULARITY LEMMA

For a graph $G$ let $V(G)$ denote the set of vertices of $G$, let $E(G)$ stand for the set of its edges and $e(G) = |E(G)|$. Let $A$, $B$ are disjoint subsets of $V(G)$, then $e(A, B) = e_G(A, B)$ denotes the number of edges $\{v, w\}$ with $v \in A$ and $w \in B$. We say that a pair $(A, B)$, where $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, is $(\varepsilon, G)$-regular for some $\varepsilon > 0$ if for every $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geqslant \varepsilon\, |A|$ and $|B'| \geqslant \varepsilon\, |B|$, we have

$$\left| \frac{e(A', B')}{|A'|\, |B'|} - \frac{e(A, B)}{|A|\, |B|} \right| < \varepsilon.$$

A partition $\Pi = (V_i)_{i=0}^k$ of the vertex set $V(G)$ of $G$ is $(\varepsilon, k)$-equitable if $|V_0| \leqslant \varepsilon\, |V(G)|$ and $|V_1| = \cdots = |V_k|$. An $(\varepsilon, k)$-equitable partition $\Pi = (V_i)_{i=0}^k$ is $(k, \varepsilon, G)$-regular if at most $\varepsilon \binom{k}{2}$ pairs $(V_i, V_j)$ with $1 \leqslant i < j \leqslant k$ are not $(\varepsilon, G)$-regular. Szemerédi's Regularity Lemma [7] states that every graph $G$ admits an $(k, \varepsilon, G)$-regular partition, where $k$ can be bounded independently from the choice of $G$. In the paper we use the following version of this result (for an extensive survey on different variants of the Regularity Lemma see [6]).

LEMMA 2. *For every $\varepsilon > 0$ and $k_0$ there exists $K_0 = K_0(\varepsilon, k_0) \geqslant k_0$ such that the following holds. For all graphs, $G_1$, $G_2$ and $G_3$, where $V(G_1) = V(G_2) = V(G_3) = V$ and $|V| \geqslant k_0$, there exists a partition $\Pi = (V_0, V_1, ..., V_k)$ of $V$ such that $k_0 \leqslant k \leqslant K_0$ and $\Pi$ is $(k, \varepsilon, G_s)$-regular for $s = 1, 2, 3$.* ∎

In our argument we use also some simple properties of $(\varepsilon, G)$-regular pairs. For a graph $G$ and $A \subseteq V(G)$ by the neighbourhood of $A$ we mean the set

$$N(A) = N_G(A) = \{v \in V(G) \backslash A : \{v, w\} \in E(G) \text{ for some } w \in A\},$$

and the degree $d(v)$ of a vertex $v \in V(G)$ is defined as

$$d(v) = d_G(v) = |N_G(v)| = |N_G(\{v\})|.$$

CLAIM 3. *Let $G$ be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = n \geqslant 45$. Furthermore, let $e_G(V_1, V_2) \geqslant |V_1|\, |V_2|/4$ and let the pair $(V_1, V_2)$ be $(\varepsilon, G)$-regular for $0 < \varepsilon < 0.01$. Then, for every $\ell$,*

*$1 \leqslant \ell \leqslant n - 5\varepsilon n$, and every pair of vertices $v' \in V_1$, $v'' \in V_2$, where $d(v')$, $d(v'') \geqslant n/5$, G contains a path of length $2\ell + 1$ connecting $v'$ and $v''$.*

*Proof.* Let us consider first the case when $1 \leqslant \ell \leqslant n/6$. One can easily see that the $(\varepsilon, G)$-regularity of $(V_1, V_2)$ implies the existence of $\hat{V}_1 \subseteq V_1$, $\hat{V}_2 \subseteq V_2$, such that $|\hat{V}_1| = |\hat{V}_2| \geqslant (1-\varepsilon)\, n$, $v' \in \hat{V}_1$, $v'' \in \hat{V}_2$, and the subgraph $H$ induced in $G$ by $\hat{V}_1 \cup \hat{V}_2$ has minimum degree larger than $n/5 - 2\varepsilon n$. Construct greedily a path $P_{2\ell-2} = v_0 v_1 \cdots v_{2\ell-2}$ of length $2\ell - 2$ such that $v_0 = v'$ and $V(P_{2\ell-2}) \subseteq \hat{V}_1 \cup \hat{V}_2 \backslash \{v''\}$. Note that

$$|N_H(v_{2\ell-2}) \backslash (V(P_{2\ell-2}) \cup \{v''\})| \geqslant n/5 - 2\varepsilon n - n/6 - 1 > \varepsilon n$$

and similarly

$$|N_H(v'') \backslash V(P_{2\ell-2})| > \varepsilon n,$$

so $G$ contains an edge $\{v_{2\ell-1}, v_{2\ell}\}$ joining $N_H(v_{2\ell-2}) \backslash (V(P_{2\ell-2}) \cup \{v''\})$ and $N_H(v'') \backslash V(P_{2\ell-2})$. Thus, $P_{2\ell+1} = v_0 \cdots v_{2\ell} v''$ is the path we are looking for.

Now suppose that $n/6 \leqslant \ell \leqslant n - 5\varepsilon n$ and we have already constructed a path $P_{2\ell-1} = v_0 v_1 \cdots v_{2\ell-1}$ of length $2\ell - 1$ connecting $v' = v_0$ and $v'' = v_{2\ell-1}$. Note that because the pair $(V_1, V_2)$ is $(\varepsilon, G)$-regular, all except no more than $\varepsilon n$ vertices of $V(P_{2\ell-1}) \cap V_1$ have in $V_2 \backslash V(P_{2\ell-1})$ at least $\varepsilon n$ neighbours and similarly, all except at most $\varepsilon n$ vertices of $V(P_{2\ell-1}) \cap V_2$ have at least $\varepsilon n$ neighbours in $V_1 \backslash V(P_{2\ell-1})$. Thus, we can use the $(\varepsilon, G)$-regularity of $(V_1, V_2)$ once again to infer that there exist $i$, $0 \leqslant i \leqslant 2\ell - 2$, and $w', w'' \in (V_1 \cup V_2) \backslash V(P_{2\ell-1})$, such that $\{v_i, w'\}$, $\{w', w''\}$ and $\{w'', v_{i+1}\}$ are edges of $G$. Consequently, the path $v_0 v_1 \cdots v_i w' w'' v_{i+1} \cdots v_{2\ell-1}$ has length $2\ell + 1$ and connects $v'$ and $v''$. ∎

The following simple consequence of the above result is crucial for our argument.

Claim 4. *Let $0 < \varepsilon < 0.001$ and let $r$ be a natural number. Furthermore, let $V_1, ..., V_{2r+1}$ be disjoint subsets of vertices of a graph G, such that $|V_1| = |V_2| = \cdots = |V_{2r+1}| = m \geqslant 90$ and for each $i = 1, ..., 2r+1$, the pair $(V_i, V_{i+1})$ is $(\varepsilon, G)$-regular and $e(V_i, V_{i+1}) \geqslant m^2/3$, where the sums of subindices are taken modulo $2r+1$. Then for each $\ell$, where $4r \leqslant \ell \leqslant 2r(1 - 15\varepsilon)\, m$, G contains a cycle of length $\ell$.*

*Proof.* Let us assume first that $\ell$ is odd. One can easily observe that there exists a cycle $C_{2r+1} = v_1 \cdots v_{2r+1} v_1$ such that for every $i = 1, 2, ..., 2r+1$ we have $v_i \in V_i$ and $v_i$ has more than $m/5$ neighbours in each of the sets $V_{i-1}$ and $V_{i+1}$. Now it is enough to apply Claim 3 and enlarge $C_{2r+1}$ using edges which join $V_{2j}$ and $V_{2j+1}$ for some $j = 1, ..., r$.

If $\ell$ is even split each of the sets $V_i$ into $V_i'$ and $V_i''$, where $|V_i'|, |V_i''| \geqslant \lfloor m/2 \rfloor$. Then one can apply a similar argument as above to expand a cycle $C_{4r}$ which has precisely one vertex of large degree in each of the sets of the "fat" cycle $V_1' V_2' \cdots V_{2r+1}' V_{2r}'' V_{2r-1}'' \cdots V_3'' V_2'' V_1'$. (Observe that the density of the bipartite graph spanned by $V_i'$ and $V_{i+1}'$ can be slightly smaller than 1/3, but, due to the $(\varepsilon, G)$-regularity of $(V_i, V_{i+1})$, it is larger than $1/3 - \varepsilon > 1/4$). ∎

*Remark.* Note that if $\ell$ is even then one can construct a cycle of length $\ell$ in $G$ which contains no edges joining $V_1$ and $V_{2r+1}$, i.e., the existence of a long even cycle of length $\ell$ in $G$ follows from the fact that $G$ contains a long "fat" path $V_1 \cdots V_{2r+1}$.

## 2. GRAPHS WITHOUT LONG ODD CYCLES

Let us recall that a classical result of Erdős and Gallai [3] states that each graph contains a cycle at least as long as the average degree of its vertices, i.e., the following holds.

LEMMA 5. *Each graph with n vertices and at least $(m-1)(n-1)/2+1$ edges, where $3 \leqslant m \leqslant n$, contains a cycle of length at least m.*

We would like to characterize the structure of graphs which contain no long cycles of odd length. Let us start with the following observation.

LEMMA 6. *Let $\delta$ be a constant such that $0 < \delta \leqslant 10^{-5}$ and let n, m be natural numbers for which*

$$\exp(\delta^{-15}) \leqslant m \leqslant n \leqslant m/\delta.$$

*Then in each graph G with n vertices and at least $\delta n^2/2$ edges which contains no odd cycles of length larger than m one can find an induced subgraph H with at least $\delta m/2$ vertices such that:*

(i) *there are at most $\delta^4 n\, |V(H)|/2$ edges between $V(H)$ and $V(G) \backslash V(H)$,*

(ii) *H is either bipartite or it contains at most $(1+\delta^4)\, m\, |V(H)|/2$ edges.*

*Proof.* We split the proof into the following two cases.

*Case* 1. $G$ contains no odd cycles of length larger than $\delta m/2$.

Since $|E(G)| > \delta n^2/2$, Lemma 5 implies that $G$ contains a cycle $C_\ell = v_1 v_2 \cdots v_\ell v_1$, where $\ell \geqslant \delta n \geqslant \delta m$ and, because of our assumption, $\ell$ must be even. Set $N_0^{\geqslant 2} = V(C_\ell)$, and for $r \geqslant 1$ let $N_r^{\geqslant 2}$ denote the set of all vertices from $V(G) \setminus \bigcup_{i=0}^{r-1} N_i^{\geqslant 2}$ which have at least two different neighbours in $N_{r-1}^{\geqslant 2}$. Note that since all sets $N_i^{\geqslant 2}$, $0 \leqslant i \leqslant r-1$, are disjoint there exists $r_0$, $1 \leqslant r_0 \leqslant \lceil 3/\delta^4 \rceil$, such that $|N_{r_0}^{\geqslant 2}| \leqslant \delta^4 n/3$. We shall show that one can take as $H$ the subgraph induced in $G$ by $\bigcup_{i=0}^{r_0-1} N_i^{\geqslant 2}$.

Note first that the number of edges between $V(H)$ and $V(G) \setminus V(H)$ is bounded from above by

$$n(r_0+1) + |N_{r_0}^{\geqslant 2}|\, |V(H)| \leqslant 5n/\delta^4 + \delta^4 n\, |V(H)|/3 \leqslant \delta^4 n\, |V(H)|/2.$$

Hence, it is enough to prove that $H$ is bipartite.

Let us suppose that, on the contrary, $H$ contains an odd cycle $C'$. Let $w' \in N_{r'}^{\geqslant 2}$ and $w'' \in N_{r''}^{\geqslant 2}$ be two different vertices of $C'$ chosen in such a way that $r'$ and $r''$ are as small as possible. Then there exists a path $P_{r'}$ of length $r'$ which connects $w'$ with some vertex $v'$ on $C_\ell$, and another path $P_{r''}$ of length $r''$, vertex-disjoint with $P_{r'}$, which joins $w''$ to some $v'' \in V(C_\ell)$. One can easily check that this implies the existence of a path $P_s$ of length $s$ connecting vertices $v_i$ and $v_j$ of $C_\ell$, $1 \leqslant i < j \leqslant \ell$, which is such that $E(P_s) \subseteq (E(C') \cup E(P_{r'}) \cup E(P_{r''})) \setminus E(C_\ell)$ and $s$ and $j-i$ (and thus also $l-(j-i)$) are of different parity. Hence, the graph induced by $E(C_\ell) \cup E(P_s)$ contains two odd cycles of which at least one is longer than $\ell/2 \geqslant \delta m/2$. Thus, since in $G$ there are no odd cycles of such length, $H$ must be bipartite.

*Case* 2. $G$ contains odd cycles longer than $\delta m/2$.

Now let $C_\ell = v_1 v_2 \cdots v_\ell v_1$ be the longest odd cycle in $G$, where, due to our assumptions, $\delta m/2 \leqslant \ell \leqslant m$. Furthermore, let $W_\ell$ denote the set of all vertices of $V(G) \setminus V(C_\ell)$ which have more than $\delta^4 \ell/8$ neighbours in $C_\ell$. We shall show that one can take as $H$ the subgraph induced in $G$ by $V(C_\ell) \cup W_\ell$.

If $|W_\ell| \leqslant \delta^4 \ell/3$ then the number of edges in $H$ can be crudely bounded from above by

$$\binom{|V(H)|}{2} \leqslant (1+\delta^4/3)\,\ell\, |V(H)|/2 \leqslant (1+\delta^4)\, m\, |V(H)|/2,$$

and the number of edges with one end in $V(H)$ is less than

$$n\delta^4 \ell/8 + n\, |W_\ell| < n\delta^4 \ell/2.$$

Thus, we may restrict ourselves to the case when $|W_\ell| > \delta^4 \ell/3$.

Note first that the subgraph $H'$ spanned by $W_\ell$ is rather sparse, namely it contains no more than $\delta^6\, |W_\ell|^2/4$ edges. Indeed, suppose that it is not the case. Then, Lemma 5 implies that $H'$ contains a path $P$ of length at least $\delta^6\,|W_\ell|/4 \geqslant 200\delta^{-18}$. Thus, let $w_1, ..., w_r$, $r = \lceil 17\delta^{-4} \rceil$, be vertices of $W_\ell$ such that for $i = 1, 2, ..., r-1$ the distance in $P$ between $w_i$ and $w_{i+1}$ is an even number larger than $7\delta^{-14}$. Note that at least $\delta^4 \ell/16$ vertices from $C_\ell$ are neighbours of at least two vertices from $w_1, ..., w_r$, so among $w_1, ..., w_r$, one can find two vertices, say $w_{i'}$ and $w_{i''}$, with at least $\delta^4 \ell/16r^2 \geqslant \delta^{14}\ell$ common neighbours in $C_\ell$. Thus, there exist three such neighbours, say $v', v'', v''' \in V(C_\ell)$, such that $v'$ and $v''$, as well as $v''$ and $v'''$, lie within distance $3\delta^{-14}$ on $C_\ell$. Hence the distance between either $v'$ and $v''$, or $v''$ and $v'''$, or maybe between $v'$ and $v'''$, is even and smaller then $6\delta^{-14}$. But then one can replace a segment of $C_\ell$ between these vertices using the segment of the path $P$ which lies between $w_{i'}$ and $w_{i''}$ and construct an odd cycle longer than $\ell$ contradicting the choice of $C_\ell$. Consequently, $H'$ must have less than $\delta^6\,|W_\ell|^2/4$ edges.

Note also that the number of edges between $W_\ell$ and $V(C_\ell)$ is not larger than $|W_\ell|(\ell+3)/2+\ell$. Indeed, otherwise one could find vertices $x_1, x_2 \in W_\ell$ and $y_1, z_1, y_2, z_2 \in V(C_\ell)$ such that $\{y_1, z_1\} \neq \{y_2, z_2\}$, for $i = 1, 2$, vertices $y_i$ and $z_i$ are adjacent in $C_\ell$ and, finally, both $y_i$ and $z_i$ are neighbours of $x_i$. But this would clearly lead to an odd cycle of length $\ell+2$, contradicting again the choice of $C_\ell$.

Hence the number of edges of $H$ is bounded from above by

$$\binom{\ell}{2} + |W_\ell|\,\frac{\ell+3}{2} + \ell + \frac{\delta^6\,|W_\ell|^2}{4} \leqslant \frac{\ell+|W_\ell|}{2}\left(\ell+4+\frac{\delta^6\,|W_\ell|}{2}\right)$$
$$\leqslant \frac{|V(H)|}{2}\left(\ell+4+\frac{\delta^6 n}{2}\right) \leqslant \frac{|V(H)|}{2}\left(\ell+4+\frac{\delta^5 m}{2}\right) \leqslant m(1+\delta^4)\,|V(H)|/2.$$

Thus, to complete the proof one needs to estimate the number of edges with precisely one end in $V(H)$. We first show that there are at most $\delta^4 \ell n/8$ edges between $W_\ell$ and $V(G)\backslash V(H)$. Note that since these two sets are connected by at most

$$|W_\ell|\,|V(G)\backslash V(H)| \leqslant n \min\{|W_\ell|, |V(G)\backslash V(H)|\}$$

edges, it is enough to consider the case when both $W_\ell$ and $V(G)\backslash V(H)$ contain at least $\delta^4 \ell/8$ vertices.

Let us suppose that the number of edges between $W_\ell$ and $V(G)\backslash V(H)$ is larger than $\delta^4 \ell n/8$. Then, from Zarankiewicz's theorem (e.g., [4, Theorem 5.1.3]), the bipartite graph induced in $G$ by $(W_\ell, V(G)\backslash V(H))$ contains a complete bipartite subgraph $H^{s,s}$ on $2s \geqslant 7\delta^{-14}$ vertices. Now, arguing as above when we dealt with the density of the subgraph spanned by $W_\ell$, one can show that there exist two vertices $w', w'' \in V(H^{s,s}) \cap W_\ell$ and two vertices $v', v'' \in V(C_\ell)$ such that the distance between $v', v''$ in $C_\ell$ is an even number smaller than $6\delta^{-14}$, $v'$ is adjacent to $w'$, and $v''$ is joined to $w''$. Since $w'$ and $w''$ are connected by a path of length $2s \geqslant 7\delta^{-14}$ in $H^{s,s}$ this would lead to an odd cycle longer than $\ell$. Thus, between $W_\ell$ and $V(G)\setminus V(H)$ there are no more than $\delta^4 \ell n/8$ edges.

Consequently, the number of edges joining $V(H)$ and $v(G)\setminus V(H)$ is bounded from above by

$$\frac{\delta^4 \ell n}{8} + \frac{\delta^4 \ell n}{8} \leqslant \frac{\delta^4 \ell n}{4}$$

and the assertion follows. ∎

Claim 7. *For every* $0 < \delta < 10^{-15}$, $\alpha > 2\delta$ *and* $n \geqslant \exp(\delta^{-16}/\alpha)$ *the following holds. Each graph* $G$ *on* $n$ *vertices which contains no odd cycles longer than* $\alpha n$ *contains subgraphs* $G'$ *and* $G''$ *such that*:

(i) $V(G') \cup V(G'') = V(G)$, $V(G') \cap V(G'') = \varnothing$ *and each of the sets* $V(G')$ *and* $V(G'')$ *is either empty or contains at least* $\alpha\delta n/2$ *vertices*;

(ii) $G'$ *is bipartite*;

(iii) $G''$ *contains not more than* $\alpha n |V(G'')|/2$ *edges*;

(iv) *all except no more than* $\delta n^2$ *edges of* $G$ *belong to either* $G'$ *or* $G''$.

*Proof.* Apply recursively Lemma 6 to find a decomposition of $G$ into graphs $H_1, H_2, ..., H_r$ such that for each $i = 1, ..., r$

(i) $|V(H_i)| \geqslant \alpha\delta n/2$;

(ii) there are at most $\delta^4 n\, |V(H_i)|/2$ edges between $V(H_i)$ and $\bigcup_{j \geqslant i+1}^{r} V(H_j)$;

(iii) $H_i$ is either bipartite or it contains at most $\alpha n(1+\delta^4)|V(H_i)|/2$ edges.

Now take as $G'$ the sum of those graphs among $H_1, ..., H_r$ which are bipartite, put all remaining vertices into $G''$, and remove the "surplus" of $e(G'') - \alpha n\, |V(G'')|/2$ edges from $G''$. ∎

In particular, from Claim 7 it follows that a dense graph without long odd cycles contains a large induced subgraph which is "almost" bipartite. Our next observation is a simple consequence of this fact.

Claim 8. *Let* $0 < \eta < 10^{-5}$ *and* $n \geqslant \exp(\eta^{-49})$. *Furthermore, let* $G$ *be a graph with* $(2+\eta)\, n$ *vertices and at least* $\binom{|V(G)|}{2} - 14\eta^3 n^2$ *edges. Then every 2-colouring of the edges of* $G$ *leads to a monochromatic odd cycle of length at least* $(1+\eta/10)\, n$.

*Proof.* Let $(G_1, G_2)$ be a 2-edge colouring of $G$ and let $|E(G_1)| \geqslant |E(G_2)|$. Suppose that $G_1$ contains no odd cycles longer than $(1+\eta/10)\, n-1$. Apply Claim 7 with $\delta=\eta^3$ and find in $G_1$ a bipartite subgraph $G_1'$ and a "sparse" subgraph $G_1''$. Since the average degree of vertices of $G_1$ is at least $(1+\eta/3)\, n$, while the average degree of vertices of $G_1''$ is at most $(1+\eta/10)\, n$, the average degree of vertices of $G_1'$ must be at least as large as $(1+\eta/4)\, n$. Thus, at least one set $S$ of the bipartition of $G_1'$ is larger than $(1+\eta/4)\, n$. Since all, except at most $(2+\eta)^2\, \eta^3 n^2$, edges of $G$ inside $S$ belong to $G_2$, and there are at least

$$\binom{|S|}{2} - (2+\eta)^2\, \eta^3 n^2 - 14\eta^3 n^2 \geqslant (1-40\eta^3) \binom{|S|}{2}$$

of them, there exists a subset $S'$ of $S$, $|S'| \geqslant (1+\eta/5)\, n$, such that $S'$ contains an odd number of vertices and the subgraph $H$ spanned by $S'$ in $G_2$ has minimum degree larger than $|S'|/2$. A hamiltonian cycle of $H$ whose existence follows from Dirac's theorem is the long odd monochromatic cycle we are looking for. ∎

Now we state and prove the main result of this section.

Lemma 9. *For every $0<\eta<10^{-5}$ and $n \geqslant \exp(\eta^{-50})$ the following holds. If $G$ is a graph with $N=4(1+\eta)\, n$ vertices and at least $(1-\eta^3)\binom{N}{2}$ edges, then each 3-colouring of the edges of $G$ leads to a monochromatic odd cycle of length at least $(1+\eta/10)\, n$.*

*Proof.* Let $(G_1, G_2, G_3)$ be a 3-edge colouring of $G$, where $|E(G_1)| \geqslant |E(G)|/3$. If $G_1$ contains no odd cycle of length at least $(1+\eta/10)\, n$, then one can apply Claim 7 with $\delta=\eta^3$ and find in $G_1$ a bipartite subgraph $G_1'$ and a sparse subgraph $G_1''$. Let $V_1=V(G_1'')$ and $V_2$ and $V_2'$ be sets of the bipartition of $G_1'$. Furthermore, let $\hat{G}$ be the graph which consists of all edges of $G_2$ and $G_3$ which either connect $V_1$ and $V_2 \cup V_2'$, or are contained in one of the sets $V_2$ or $V_2'$. Thus, in order to prove the Lemma it is enough to show that every 2-colouring of the edges of $\hat{G}$ leads to a long monochromatic odd cycle, i.e., either the subgraph $\hat{G}_2$ with edges $E(G_2) \cap E(\hat{G})$, or $\hat{G}_3$ spanned by $E(G_3) \cap E(\hat{G})$ contains an odd cycle of length at least $(1+\eta/10)\, n$.

Let us put $|V_1|=\lambda N$ and $|V_2 \cup V_2'|=(1-\lambda)\, N$. Then, from the fact that

$$|E(G_1')| + |E(G_1'')| \geqslant |E(G_1)| - \eta^3 N^2 \geqslant \frac{1-\eta^3}{3} \binom{N}{2} - \eta^3 N^2$$

we get

$$\frac{(1-\lambda)^2}{4}+\frac{\lambda(1+\eta/9)}{8(1+\eta)}\geqslant\frac{1}{6}-2\eta^3,$$

from which it follows that $\lambda<0.28$. Furthermore, note that if either $V_2$ or $V_2'$ has more than $(2+\eta)\,n=N/2-\eta n$ vertices, then the assertion follows from Claim 8. Thus, we may and shall assume that $|V_1|\geqslant\eta n$.

Clearly, we may also assume that

$$|E(\hat{G}_2)|\geqslant\frac{|E(\hat{G})|}{2}\geqslant\frac{(1-\lambda)(1+3\lambda)}{8}N^2-2\eta^3N^2.$$

If $\hat{G}_2$ contains no odd cycles of length at least $(1+\eta/10)\,n$ we can use Claim 7 to find in $\hat{G}_2$ a bipartite subgraph $\hat{G}_2'$ and a sparse $\hat{G}_2''$. Put $|V(\hat{G}_2')|=\mu N$ and $|V(\hat{G}_2'')|=(1-\mu)\,N$. Then, as in the case of $\lambda$, we must have

$$\frac{\mu^2}{4}+\frac{(1-\mu)(1+\eta/9)}{8(1+\eta)}\geqslant\frac{(1-\lambda)(1+3\lambda)}{8}-2\eta^3-\eta^3.$$

Using the fact that $\eta\leqslant 10^{-5}$ and $\lambda<0.28$, after simple calculations one can check that the above inequality implies that

$$\mu\geqslant\frac{1}{4}+\frac{1}{4}\sqrt{1+5\eta+16\lambda-24\lambda}>\frac{1}{2}(1+\eta)+0.8\lambda. \tag{1}$$

Now denote by $W_1$ and $W_2$ the two sets of the bipartition of $\hat{G}_2'$. The rest of the proof will be based on the following simple idea. Let us consider, say, the set $W_1\cap V_2$. Note that all, except at most $3\eta^3N^2$, pairs of vertices from this set are edges of $G_3$. Consequently, for every subset $S\subseteq W_1\cap V_2$ one can find $S'\subseteq S$ with $|S'|>|S|-\eta n/10$ such that the subgraph induced in $G_3$ by $S'$ has minimum degree at least $|S'|/2+2$ and thus, from Dirac's theorem, remains Hamiltonian-connected even if we delete from it one vertex to adjust the parity of the Hamiltonian path. Similarly, all except no more than $3\eta^3N^2$ pairs $\{v,w\}$ such that, say, $v\in V_1\cap W_1$ and $w\in(V_2\cup V_2')\backslash W_2$, belong to $E(G_3)$. Thus, if we choose $S\subseteq V_1\cap W_1$, $T\subseteq(V_2\cup V_2')\backslash W_2$ such that $|S|=|T|$ then each pair of two different vertices of $S\cup T$ is connected by a path $P$ for which $E(P)\subseteq\{\{s,t\}:s\in S,\ t\in T\}$ and $|V(P)|\geqslant|S\cup T|-\eta n/10$ (clearly the parity of $|V(P)|$ depends only of the choice of the ends of $P$). Below we shall show that $G_3$ contains an odd cycle of length at least $(1+\eta/10)\,n$ obtained by merging together long paths contained in $W_1\cap V_2$, $W_1\cap V_2'$, $W_2\cap V_2$ and $W_2\cap V_2'$ and paths which consist of edges joining $W_1$ and $(V_1 \cup V_2 \cup V_2')\backslash W_2$, and those between $W_2$ an $(V_1 \cup V_2 \cup V_2')\backslash W_1$.

Let us suppose that $|W_1 \cap V_1| \geqslant |W_2 \cap V_1|$. We split the argument into several cases.

*Case* 1. $|W_1 \cap V_1| \leqslant 0.4\, |V_1|$.

Since $|V_1| \geqslant \eta n$, we have

$$|V_1 \backslash (W_1 \cup W_2)| \geqslant \eta n/5.$$

Hence, using vertices from $V_1 \backslash (W_1 \cup W_2)$ we can merge together large cycles contained in $W_1 \cap V_2$, $W_1 \cap V_2'$, $W_2 \cap V_2$ and $W_2 \cap V_2'$ into one odd cycle of length at least

$$|(W_1 \cup W_2) \cap (V_2 \cup V_2')| - \eta n \geqslant (1+\eta)\, N/2 + 0.8\lambda N - 0.8\lambda N - \eta n > 2n.$$

*Case* 2. $0.4\, |V_1| \leqslant |W_1 \cap V_1| \leqslant |V_1| - \eta n/2$.

One can use vertices of $V_1 \cap W_1$ and create an odd cycle $C^1$ of length at least $|W_1 \cap (V_2 \cup V_2')| - \eta n$. Similarly, we can employ vertices from $V_1 \backslash W_1$ and build an odd cycle $C^2$ of length at least $|W_2 \cap (V_2 \cup V_2')| - \eta n$. Now consider two subcases.

(1) $|(V_2 \cup V_2')\backslash (W_1 \cup W_2)| \geqslant \eta n$. Use vertices from $(V_2 \cup V_2')\backslash (W_1 \cup W_2)$ to merge $C^1$ and $C^2$ into an odd cycle, which, due to (1) and the fact that $\lambda < 0.28$, is longer than

$$\begin{aligned} |(V_2 \cup V_2') \cap (W_1 \cup W_2)| - 2\eta n &\geqslant (1+\eta)\, N/2 + 0.8\lambda N - 2\eta n - \lambda N \\ &> 0.44N > 1.5n. \end{aligned}$$

(2) $|(V_2 \cup V_2')\backslash (W_1 \cup W_2)| \leqslant \eta n$. From (1) and the upper bound for $|V_1|$ we have

$$|V(C^1)| + |V(C^2)| \geqslant |(V_2 \cup V_2') \cap (W_1 \cup W_2)| - 2\eta n \geqslant N - \lambda N - 3\eta n \geqslant 2.3n.$$

Thus, at least one of the cycles $C^1$ and $C^2$ must be longer than $1.1n$.

*Case* 3. $|V_1 \backslash W_1| \leqslant \eta n/2$ and $|V_1| = \lambda N \geqslant N/8$.

Here we have a few possibilities. If $|(V_2 \cup V_2')\backslash W_2| \geqslant N/8$ and $|W_1 \cap (V_2 \cup V_2')| \geqslant \eta n$ then one can build an odd cycle of length at least $N/4 - \eta n/2 > (1+\eta/10)\, n$ using only edges of $G_3$ joining $V_1 \cap W_1$ and $(V_2 \cup V_2')\backslash W_2$ and one edge contained in $W_1 \cap (V_2 \cup V_2')$. Thus, let us suppose that $|(V_2 \cup V_2')\backslash W_2| \geqslant N/8$ but $|W_1 \cap (V_2 \cup V_2')| \leqslant \eta n$ and let $|V_2| \geqslant |V_2'|$, i.e.,

$$|V_2| \geqslant N(1-\lambda)/2 > 0.36N.$$

If $|W_2 \cap V_2| \geqslant N/8$ we are done: one can create an odd cycle of length $N/4 - \eta n/2 > (1+\eta/10)\,n$ inside $V_2$ using edges of $G_3$ which have at least one end in $W_2$. Thus, suppose that $|V_2 \cap W_2| < N/8$. But then from our assumption we get

$$\begin{aligned}|W_2 \cap V_2'| &\geqslant |W_1 \cup W_2| - |V_1| - |W_2 \cap V_2| - \eta n\\ &\geqslant (1+\eta)\,N/2 + 0.8\lambda N - \lambda N - N/8 - \eta n > 0.3N > 1.2n,\end{aligned}$$

and the subgraph spanned in $G_3$ by $W_2 \cap V_2'$ contains an odd cycle longer than $1.1n$.

Finally, let $|(V_2 \cup V_2')\backslash W_2| < N/8$. Then

$$|W_2 \cap V_2| + |W_2 \cap V_2'| \geqslant N - \lambda N - N/8 > 0.55N > 2.1n.$$

Hence, at least one of the sets $W_2 \cap V_2$ and $W_2 \cap V_2'$ is larger than $1.05n$ which implies that $G_3$ contains an odd cycle longer than $(1+\eta)\,n$.

*Case* 4. $\lambda < 1/8$, $|V_1 \backslash W_1| \leqslant \eta n/2$ and $|(V_2 \cup V_2')\backslash(W_1 \cup W_2)| \leqslant \eta n$.

If $|W_1 \cap (V_2 \cup V_2')| \geqslant N/4$ then there exists an odd cycle of $G_3$ longer than $N/4 - \eta n/2 > (1+\eta/10)\,n$ contained in $W_1$, so let us assume that it is not the case, i.e., $|W_1 \cap (V_2 \cup V_2')| < N/4$. But then we have

$$\begin{aligned}|W_2 \cap (V_2 \cup V_2')| &\geqslant N - \lambda N - |(V_2 \cup V_2')\backslash(W_1 \cup W_2)| - |W_1 \cap (V_2 \cup V_2')|\\ &> N(1 - \lambda - \eta - 1/4) > 0.6N > 2.3n.\end{aligned}$$

Hence at least one of the sets $|W_2 \cap V_2|$ and $|W_2 \cap V_2'|$ is larger than $1.15n$ and thus it spans in $G_3$ the subgraph which contains an odd cycle longer than $1.1n$.

*Case* 5. $\lambda < 1/8$, $|V_1 \backslash W_1| \leqslant \eta n/2$ and $|(V_2 \cup V_2')\backslash(W_1 \cup W_2)| \geqslant \eta n$.

Note that if $\lambda < 1/8$ then (1) becomes

$$\mu \geqslant \tfrac{1}{4} + \tfrac{1}{4}\sqrt{1 + 5\eta + 16\lambda - 24\lambda} > \tfrac{1}{2}(1+\eta) + 1.2\lambda$$

and thus

$$|(W_1 \cup W_2) \cap (V_2 \cup V_2')| \geqslant (1+\eta)\,N/2 > 2(1+\eta)\,n. \qquad (2)$$

Now let $|V_2 \backslash (W_1 \cup W_2)| \geqslant \eta n/2$. Then, using vertices from $V_2 \backslash (W_1 \cup W_2)$ we can create an odd cycle $C^3$ of length at least $|W_1 \cap (V_1 \cup V_2')| + |W_2 \cap V_2| - \eta n/2$. Furthermore, one can find an odd cycle $C^4$ of length at least $|W_2 \cap V_2'| - \eta n/2$. From (2) one of these two cycles must be longer than $(1+\eta/10)\,n$. ∎

## 3. PROOF OF THE MAIN RESULT

Theorem 1 follows rather easily from Lemma 2, Claim 4, and Lemma 9.

*Proof of Theorem* 1. Let $0<\eta<10^{-5}$. We need to show that each 3-edge colouring of $K_{4(1+\eta)\, n}$ leads to a monochromatic cycle of length $n$, provided $n$ is large enough. Set $\varepsilon=\eta^3$, $k_0=5\exp(\eta^{-50})$ and let $K_0=K_0(\varepsilon, k_0)$ be defined as in Lemma 2. We prove that the statement of Theorem 1 holds for $n \geqslant K_0$.

Consider a 3-edge colouring of $(G_1, G_2, G_3)$ of $K_{4(1+\eta)\, n}$ and apply Lemma 2 to find a partition $\Pi=(V_0, V_1, ..., V_k)$ such that $k_0 \leqslant k \leqslant K_0$ and $\Pi$ is $(k, \varepsilon, G_s)$-regular for $s=1, 2, 3$. Let $\tilde{G}$ be the graph with vertex set $\{1, 2, ..., k\}$ and set of edges defined as

$$E(\tilde{G})=\{\{i, j\}: (V_i, V_j) \text{ is } (\varepsilon, G_s)-\text{regular for } s=1, 2, 3\}.$$

Then, $e(\tilde{G}) \geqslant (1-\varepsilon)\binom{k}{2}=(1-\eta^3)\binom{k}{2}$. Define a 3-edge colouring $(\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)$ of $\tilde{G}$ be colouring an edge $\{i, j\}$ of $\tilde{G}$ with the first colour $s$ for which $e_{G_s}(V_i, V_j) \geqslant |V_i|\, |V_j|/3$. Lemma 9 implies that in such a colouring we can find a monochromatic odd cycle $\tilde{C}$ in one of the colours, say $\tilde{G}_1$, which is longer than $(1+\eta/10)\, k/4(1+\eta)$. Note that $\tilde{C}$ is a "fat" odd cycle which fulfills the assumptions of Claim 4 with $r \geqslant (1+\eta/11)\, k/8(1+\eta)$ and $m \geqslant (4(1+\eta)\, n-\eta^3 n)/k$. Consequently, $G_1$ contains a cycle of each length $\ell$, provided $\ell \geqslant K_0$ and

$$\ell \leqslant (1+\eta/12)\, n \leqslant 2\,\frac{1+\eta/11}{8(1+\eta)}\, k(1-15\eta^3)\, 4(1+\eta-\eta^3)\,\frac{n}{k} \leqslant (1+\eta/11)\, n;$$

in particular, it contains a cycle of length $n$. ∎

Let us conclude with some remarks on results one can show using a similar approach. Let $C_n^{\times k}$ denote the graph on $kn$ vertices obtained from a cycle $C_n$ by replacing each of its vertices by a set of $k$ vertices, and each edge by a complete bipartite graph on $2k$ vertices. Then, one can prove an analog of Claim 3 for a "blown-up" paths (or just apply the much stronger Blow-Up Lemma of Komlós, Sárközy and Szemerédi [4]) and use Lemma 9 to show the following.

THEOREM 10. *For a fixed $k$ we have*

$$\limsup_{n\to\infty} \frac{R(C_n^{\times k}, C_n^{\times k}, C_n^{\times k})}{n} \leqslant 4k. \quad ∎$$

Since when both $k$ and $n$ are odd $C_n^{\times k}$ contains an odd cycle $C_{kn}$, for such a case the constant $4k$ in Theorem 10 cannot be replaced by a smaller one.

In the even case, the estimates given by Theorems 1 and 10 can be easily improved. Indeed, as we have mentioned in the remark following Claim 4, the existence of a long path in $\tilde{G}$ implies that $G$ contains long even cycles of any prescribed length. Thus, one can use Lemma 9 instead of Lemma 5 and easily arrive at the following result.

*Claim* 11. For an even $n$ we have $R(C_n, C_n, C_n) \leqslant (3 + o(1))\, n$. ∎

On the other hand it seems that, in fact, for even $n$ we have

$$R(C_n, C_n, C_n) = (2 + o(1))\, n. \tag{3}$$

Note that from the proof of Theorem 1 it follows that (3) holds if, say, for every $\eta > 0$ and large enough $n$, each 3-edge colouring of a graph $G$ with $N = 2(1+\eta)\, n$ vertices and at least $(1-\eta^{20})\binom{N}{2}$ edges leads to a monochromatic path of length $(1+\eta/10)\, n$. As a matter of fact, in order to prove the above equality much less is needed; for instance, it is not hard to see that (3) would follow from a positive answer to the following problem.

*Problem.* Decide whether for every $\eta > 0$ there exists $n_0$ such that for $n \geqslant n_0$ the following holds. For every 3-edge colouring of a graph $G$ with $N = 2(1+\eta)\, n$ vertices and more than $(1-\eta^{20})\binom{N}{2}$ edges there exists a monochromatic tree containing a matching which saturates at least $(1+\eta/10)\, n$ vertices of $G$.

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